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1987 J. Phys. A: Math. Gen. 20 L1057

(http://iopscience.iop.org/0305-4470/20/16/004)

J. Phys. A: Math. Gen. 20 (1987) L1057–L1062. Printed in the UK

LETTER TO THE EDITOR

On spectral problems and compatibility conditions in multidimensions

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Received 11 June 1987

Abstract. A method for the construction of auxiliary linear problems suitable for the inverse spectral transform method is considered. An algebraic form of the compatibility conditions for these linear problems is discussed for the three-dimensional space.

The starting point of the inverse spectral transform method is the representation of the non-linear differential equation as the compatibility condition of the certain set of auxiliary linear problems (see e.g. [1]). The algebraic forms of these compatibility conditions are the well known Lax pair [2], the commutativity condition $[L_1, L_2] = L_1L_2 - L_2L_1 = 0$ [3], Manakov's L-A-B triad [4] or Zakharov's algebraic system [5]. Recently Manakov and Zakharov have proposed a new method for the construction of the multidimensional auxiliary linear problems based on the non-local Riemann conjugation problem [6].

Here we present the general formulation of this non-local Riemann problem method in the generic multidimensional case and consider the possible algebraic forms of the compatibility conditions in multidimensions.

The starting point of the Manakov–Zakharov method [6] is the non-local Riemann problem

$$\psi_2(\lambda, x) = \int_\Gamma \mathrm{d}\lambda' \, \psi_1(\lambda', x) R(\lambda', \lambda, x) \tag{1}$$

where $\lambda \in \mathbb{C}$, $x = (x_1, \ldots, x_d)$ and ψ_2, ψ_1 are boundary values of the analytic function on the contour Γ and $R(\lambda', \lambda, x)$ is the certain matrix function. It is assumed that the function R obeys the equations

$$\frac{\partial}{\partial x_i} R(\lambda', \lambda, x) = I_i(\lambda') R(\lambda', \lambda, x) - R(\lambda', \lambda, x) I_i(\lambda) \qquad i = 1, \ldots, d \tag{2}$$

where $I_i(\lambda)$ are certain matrix functions and $[I_i(\lambda'), I_k(\lambda)] = 0$. Then the operators D_i ($D_i f \overset{\text{def}}{=} \partial_{x_i} f + f I_i(\lambda)$) are introduced and with the use of (1) and (2) the set of operators L_i of the form $L_i = \Sigma_{n_i} q'_{n_1,\ldots,m_d}(x_1, \ldots, x_d) D_1^{n_1} \ldots D_d^{n_d}$ which have no singularities on λ is constructed. The compatibility of the linear system $L_i\psi = 0$ ($i = 1, \ldots, k$) is equivalent to the non-linear equation. Some concrete examples have been considered in [6].

We would like to propose a scheme which naturally leads to the conjugation problem (1) and its generalisations. We start with the formal expansion problem

$$\psi_2(\lambda, x) = \int \mathrm{d}\lambda' \psi_1(\lambda', x) R(\lambda', \lambda, x) \tag{3}$$

where $\lambda = (\lambda_1, \ldots, \lambda_d)$, $x = (x_1, \ldots, x_d)$, ψ_2 and ψ_1 are matrix functions and $R(\lambda', \lambda, x)$ is the certain matrix function. Note that all λ_i are independent variables. We assume that $R(\lambda', \lambda, x)$ satisfies the equations ($\partial_{x_i} \equiv \partial/\partial x_i$)

$$\partial_{x_i} R(\lambda', \lambda, x) = \lambda'_i R(\lambda', \lambda, x) - R(\lambda', \lambda, x)\lambda_i \qquad i = 1, \ldots, d \tag{4}$$

where $[\lambda_i, \lambda_k] = 0$. Denote $D_i f \stackrel{\text{def}}{=} \partial_{x_i} f + f\lambda_i$. The problem in which we are interested is to construct the operators L_i of the form $L_i = \Sigma_{n_1,\ldots,n_d} U'_{n_1,\ldots,n_d}(x) D_1^{n_1} \ldots D_d^{n_d}$ which have no singularities at $\lambda_i \to \infty$ ($i = 1, \ldots, d$). Similar to [6] one has $L_i(\lambda)\psi_2(\lambda x) = \int \mathrm{d}\lambda' \, L_i(\lambda')\psi_1(\lambda', x) R(\lambda', \lambda, x)$.

It is not difficult to see that it is not possible to construct such an operator L if all the variables $\lambda_1, \ldots, \lambda_d$ are independent ones. Indeed, let the highest-order terms in L be $\varphi(D_1, \ldots, D_d)$. So $Lf = \varphi(D_1, \ldots, D_d) f + f\varphi(\lambda_1, \ldots, \lambda_d) + \Delta$ where Δ contains lower-order terms in $D_1, \ldots, D_d$. The term $f\varphi(\lambda_1, \ldots, \lambda_d)$ cannot be excluded at all.

This consideration shows also that the only way to construct the operator L without singularities at $\lambda_i \to \infty$ is to impose some constraint on the variables $\lambda_1, \ldots, \lambda_d$.

Let the variables $\lambda_1, \ldots, \lambda_d$ be constrained by the algebraic equation

$$\varphi(\lambda_1, \ldots, \lambda_d) = C = \text{constant} \tag{5}$$

where $\varphi(\lambda_1, \ldots, \lambda_d)$ is some polynomial.

Proposition. If the variables $\lambda_1, \ldots, \lambda_d$ in (3) obey the constraint (5) then the operator L which has no singularity at $\lambda_i \to \infty$ is of the form $L = \varphi(D_1, \ldots, D_d) + \Delta$.

Indeed $L\psi = \varphi(D_1, \ldots, D_d)\psi + \Delta\psi = \varphi(\partial_{x_1}, \ldots, \partial_{x_d})\psi + \psi\varphi(\lambda_1, \ldots, \lambda_d) + \tilde{\Delta}$. The highest singularities on $\lambda_1, \ldots, \lambda_d$ which are collected to the term $\psi\varphi(\lambda_1, \ldots, \lambda_d)$ are annihilated due to the constraint (5). The singularities of the lower order are annihilated by the procedure described in [6].

Example. $\varphi(\lambda_1, \ldots, \lambda_d) = \lambda_1^2 + \ldots + \lambda_d^2 = \text{constant}$. The corresponding operator L is $L = \partial_{x_1}^2 + \ldots + \partial_{x_d}^2 + \Sigma_{i=1}^d U_i \partial_{x_i} + W$.

Now let us introduce the constraint

$$\varphi_\alpha(\lambda_1, \ldots, \lambda_d) = C_\alpha = \text{constant} \qquad (\alpha = 1, \ldots, n). \tag{6}$$

Thus we are able to construct n operators L_α without singularities. Let us consider the system of n linear equations

$$L_\alpha \psi = 0 \qquad \alpha = 1, \ldots, n. \tag{7}$$

A necessary condition of the compatibility of this system is the existence of a non-empty cross section of the surfaces (6). If the cross section of the surfaces (6) is empty then the system (7) has no non-trivial solution.

Let the constraints (6) be independent (i.e. all functions $\varphi_\alpha(\lambda_1, \ldots, \lambda_d)$ are algebraically independent) and their cross section S has a generic complex dimension $d - n$.

Let us parametrise (uniformise) this cross section S by $d-n$ variables $\mu_1, \ldots, \mu_{d-n}$. The common solutions of the corresponding system (7) depend on these uniformised variables μ_α. For these common solutions of (7) the relation (3) is equivalent to

$$\psi_2(\mu, x) = \int_S \mathrm{d}^{d-n}\mu' \rho(\mu')\psi_1(\mu', x)R(\mu', \mu, x) \tag{8}$$

where $\mu = (\mu_1, \ldots, \mu_{d-n})$ and $\rho(\mu')$ is the certain measure on the manifold S. One can consider (8) as the $(d-n)$-dimensional generalisation of the non-local Riemann conjugation problem (1), namely as the problem of construction of the (possibly analytic) function ψ whose boundary values ψ_1, ψ_2 on the $(d-n-1)$-dimensional surface Γ are related by (8) with the certain matrix function $R(\mu', \mu, x)$. Unfortunately the problem (8) has so far been effectively solved only in the one-dimensional case $d-n=1$ (see, e.g., [6]). In this case we arrive at (1) ($\mu_1 \equiv \lambda$) and can construct the linear system (7) by the method given in [6]. The possibility of the generalisation of (1) to the multidimensional manifolds has been discussed by Manakov.

At the three-dimensional space ($d=3$) we have two independent constraints $\varphi_\alpha(\lambda_1, \lambda_2, \lambda_3) = C_\alpha$ ($\alpha = 1, 2$) and $\dim S' = 1$ in the generic case. Let us consider a few illustrative examples.

(i) For constraints of the form

$$\begin{aligned} \varphi_1 &= A_1(\lambda_3)\lambda_1 - B_1(\lambda_3) = 0 \\ \varphi_2 &= A_2(\lambda_3)\lambda_2 - B_2(\lambda_3) = 0 \end{aligned} \tag{9}$$

where A_i, B_i are polynomial, one has two operators

$$L_i = \bar{A}_i(\partial_{x_3})\partial_{x_i} - \bar{B}_i(\partial_{x_3}) \qquad i = 1, 2 \tag{10}$$

where $\bar{A}_i(\partial_{x_3})$ and $\bar{B}_i(\partial_{x_3})$ are differential operators over x_3 such that $\bar{A}_i \underset{x\to\infty}{\to} A_i(\partial_{x_3})$ and $\bar{B}_i \underset{x\to\infty}{\to} B_i(\partial_{x_3})$. The system (9) defines the one-dimensional manifold Γ which can obviously be parametrised by the single variable λ_3 and $\lambda_1 = B_1(\lambda_3)/A_1(\lambda_3)$, $\lambda_2 = B_2(\lambda_3)/A_2(\lambda_3)$. The case of one marked variable x_3 has been considered in [5, 6].

(ii) The second example is

$$\varphi_1 = \lambda_1^2 - \sigma^2\lambda_2^2 = \text{constant} \tag{11a}$$

$$\varphi_2 = \lambda_3 + P(\lambda_1, \lambda_2) = 0 \tag{11b}$$

where $\sigma^2 = \pm 1$ and $P(\lambda_1, \lambda_2)$ is an arbitrary polynomial. The corresponding operators are

$$\begin{aligned} L_1 &= \partial_{x_1}^2 - \sigma^2\partial_{x_2}^2 + U_1\partial_{x_1} + U_2\partial_{x_2} + W \\ L_2 &= \partial_{x_3} + \bar{P}(\partial_{x_1}, \partial_{x_2}) \end{aligned} \tag{12}$$

where $U_1(x)$, $U_2(x)$, $W(x)$ are functions and $\bar{P}(\partial_{x_1}, \partial_{x_2})$ is the differential operator such that $\bar{P} \underset{x\to\infty}{\to} P(\partial_{x_1}, \partial_{x_2})$. The operators of the form (12) and corresponding hierarchies of integrable equations have been considered in [4, 5, 7-10]. The uniformised variable for (11) can be chosen as $\mu = \lambda_1 + \sigma\lambda_2$. The systems integrable by (12) and invariant under the rotations on the plane (x_1, x_2) correspond to $P = P(\lambda_1^2 - \sigma^2\lambda_2^2)$. For such integrable systems by virtue of (11a) one has $\lambda_3 = $ constant, corresponding to the trivial evolution law in the variable x_3 and to the linearisable systems. An example of such a rotationally invariant system is

$$\begin{aligned} &\frac{\partial W}{\partial x_3}+\varepsilon\Delta W+\varepsilon\sum_{k=1}^{2}U_k\partial_{x_k}W=0 \\ &\frac{\partial U_k}{\partial x_3}+2\varepsilon\partial_{x_k}W=0 \qquad k=1,2 \end{aligned} \tag{13}$$

which corresponds to

$$\begin{aligned} &L_1=\Delta+\sum_{k=1}^{2}U_k\partial_{x_k}+W \\ &L_2=\partial_{x_3}+L_1-\varepsilon W \end{aligned} \tag{14}$$

where $\Delta=\partial_{x_1}^2+\partial_{x_2}^2$, ε is a constant, $\sigma^2=-1$ and $P=\lambda_1^2+\lambda_2^2$. The system (13) is linearisable by the introduction of the gauge variable $g(x)$: $U_k=2\partial_{x_k}\ln g$ and $W=-g^{-1}\Delta g$. The system (13) is equivalent to the heat equation for g, i.e. $\partial g/\partial x_3+\varepsilon\Delta g=0$. In a similar manner one can consider instead of (11) the case $\varphi_1=\varphi_1(\lambda_1,\lambda_2)=C$.

(iii) Let λ_1, λ_2, λ_3 be the matrix-valued commuting variables and let the constraints be

$$\varphi_{ik}\psi=(\lambda_i^{(0)}-\lambda_k^{(0)})\psi\lambda_i\lambda_k+\psi(\lambda_kA_i-\lambda_iA_k)=0 \qquad i\neq k \qquad i,k=1,2,3 \tag{15}$$

where $[A_i,A_k]=0$ $(i,k=1,2,3)$ and $\lambda_i^{(0)}$ are constants. The corresponding operators L_{ik} are

$$L_{ik}=(\lambda_i^{(0)}-\lambda_k^{(0)})\partial_{x_i}\partial_{x_k}+U_{ik}^i(x)\partial_{x_i}+U_{ik}^k(x)\partial_{x_k}+W_{ik}(x) \qquad i\neq k \qquad i,k=1,2,3. \tag{16}$$

Since $\varphi_{12}\psi\lambda_3-\varphi_{13}\psi\lambda_3+\varphi_{23}\psi\lambda_1=0$ then only two of the constraints (15) are independent. Their cross section Γ is the one-dimensional one and possesses the rational uniformisation $\lambda_i=A_i/(\lambda-\lambda_i^{(0)})$ $(i=1,2,3)$, $\lambda\in\mathbb{C}$. The description of the rational curves by the quadrics has been discussed in [11].

The construction of the operators L_{ik} of the form (16) (up to the redefinition $U_{ik}^i\to(\lambda_i^{(0)}-\lambda_k^{(0)})U_{ik}^i$) have been considered in [6] commencing with the problem (1), with $I_i=A_i/(\lambda-\lambda_i^{(0)})$ and with the corresponding three-dimensional chiral-fields-type model equations ($W_{ik}=0$)

$$\frac{\partial U_{ik}^i}{\partial x_l}-U_{ik}^lU_{il}^i+U_{lk}^lU_{li}^i+U_{lk}^kU_{ik}^i=0 \tag{17}$$

where $U_{ik}^k=-[(\lambda_k^{(0)}-\lambda_i^{(0)})\partial g_k/\partial x_i+g_kA_i]g_k^{-1}$ and $g_k(x_1,x_2,x_3)$ $(k=1,2,3)$ are squared matrices. The summation over repeated indices in (15)-(17) is absent.

Now we will discuss the algebraic form of the compatibility conditions for the system (7) at $d=3$, i.e. for the system

$$L_1\psi=0 \qquad L_2\psi=0. \tag{18}$$

In the general case the sufficient condition for the compatibility of the system (18) is

$$C_1L_1+C_2L_2=0 \tag{19}$$

where C_1 and C_2 are certain non-degenerate differential operators or equivalently

$$[L_1, L_2] = B_1 L_1 + B_2 L_2 \tag{20}$$

where $B_1 = -C_1 - L_2$ and $B_2 = L_1 - C_2$. So in the general case the algebraic form of the compatibility condition is the quartet representations (19) or (20). In the particular case $B_2 = 0$ we arrive at the Manakov triad representation. Finally, at $B_1 = B_2 = 0$ we have the usual commutativity condition $[L_1, L_2] = 0$ [1].

Examples of non-linear systems representable in the form (19) and (20) are systems which have operators L_1, L_2 of the form (10) ($\bar{A}_i \neq 1$). In the case $C_1 = D_1(\partial_{x_3})\partial_{x_2} - F_1(\partial_{x_3})$, $C_2 = D_2(\partial_{x_3})\partial_{x_1} - F_2(\partial_{x_3})$, where D_i, F_i are differential operators over x_3, equation (19) is equivalent to the system of algebraic compatibility conditions found in [5] ($x_3 \to x$, $x_1 \to y$, $x_2 \to t$).

An example of a system representable in the form (19) is

$$\begin{gathered}(U - V)\varphi_{yt} = V_y\varphi_t - U_t\varphi_y \\ \left(\frac{(V-\varphi)_y}{U - V}\right)_x = V_y \qquad \left(\frac{(U-\varphi)_t}{U - V}\right)_x = U_t\end{gathered} \tag{21}$$

which has been found by Zakharov [5], for which

$$L_1 = \partial_x\partial_y - U\partial_y - \varphi_y \qquad L_2 = \partial_x\partial_t - V\partial_t - \varphi_t$$

and

$$\begin{gathered}C_1 = (U - V)\partial_x\partial_t + (V^2 - UV + U_x - V_x)\partial_t + (U - V)(U_t - \varphi_t) \\ C_2 = -(U - V)\partial_x\partial_y + (U^2 - UV - U_x + V_x)\partial_y - (U - V)(V_y - \varphi_y).\end{gathered} \tag{22}$$

For operators of the form (12) we have the Manakov triad representation [4, 6].

Equation (19) is invariant under the 'orthogonal' transformations

$$C_i \to C_i' = \sum_{i=1}^{2} C_k Q_{ki} \qquad L_i \to L_i' = \sum_{k=1}^{2} \bar{Q}_{ik} L_k \qquad i = 1, 2 \tag{23}$$

where Q_{ki} and $\bar{Q}_{ik}$ are arbitrary differential operators which obey the constraint $\Sigma_{i=1}^{2} Q_{ki}\bar{Q}_{il} = \delta_{kl}$. So the non-linear integrable system representable in the form (19) possesses infinitely many operators L_i and C_i defined up to the transformations (23). The uncertainty (23) can be used for the choice of appropriate operators L_1, L_2. At the case $C_2 = L_1$ ($B_2 \equiv 0$) (Manakov triad representation) the transformations (23) are reduced to the transformations $L_2 \to L_2' = L_2 + Q_{21}L_1$ and $C_1 \to C_1' = C_1 - L_1 Q_{21}$ considered in [12]. The use of this uncertainty allows one [10] to prove the existence of the matrix commutativity representations $[\bar{L}_1, \bar{L}_2] = 0$ in addition to the known Manakov triad representation for the integrable systems considered in [7-9]. This is also valid for the systems of the type (12) with $L_1 = \Sigma_{n,m=0}^{n+m=N} U_{nm}(x_1, x_2, x_3)\partial_{x_1}^n\partial_{x_2}^m$ [10].

For systems which contain the variables x_1, x_2, x_3 more symmetrically, condition (19) can be also represented in an equivalent more symmetric form. For example, for the three-dimensional chiral-fields-type equations (17) associated with the operators L_{ik} (16) (divided by $(\lambda_i^{(0)} - \lambda_k^{(0)})$ the compatibility condition is equivalent to the system of operator equalities

$$\begin{gathered}[L_{ik}, L_{nk}] = \alpha_{ink}L_{in} + \beta_{ink}L_{ik} + \gamma_{ink}L_{nk} \\ i, k, n = 1, 2, 3 \qquad i \neq k \qquad i \neq n \qquad n \neq k\end{gathered} \tag{24}$$

where there is no summation over repeated indices and

$$\begin{aligned}
\alpha_{ink} &= \frac{\partial U^n_{nk}}{\partial x^k} - \frac{\partial U^i_{ik}}{\partial x^i} + [U^i_{ik}, U^n_{nk}] \\
\beta_{ink} &= \frac{\partial U^k_{nk}}{\partial x^i} - \frac{\partial U^i_{ik}}{\partial x^n} + [U^i_{ik}, U^k_{nk}] \\
\gamma_{ink} &= \frac{\partial U^n_{nk}}{\partial x^i} - \frac{\partial U^k_{ik}}{\partial x^k} + [U^k_{ik}, U^n_{nk}].
\end{aligned} \tag{25}$$

Note that non-commutative algebraic representations of the compatibility conditions different from (20) and (24) have been considered in the other contexts in [13, 14].

I am grateful to V E Zakharov and S V Manakov for useful discussions.

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